An Introduction to p-adic Dynamics

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General Discrete Dynamics

X a set, $\phi : X \to X$

$$\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$$

The **(forward) orbit** of $x \in X$ is $\{\phi^n(x)\}_{n \geq 0}$.

The **backward orbit** of x is $\bigcup_{n \geq 0} \phi^{-n}(x)$.

We say $x \in X$ is

- a **fixed point** if $\phi(x) = x$.
- a **periodic point** if $\phi^n(x) = x$ for some $n \geq 1$.
- a **preperiodic point** if $\phi^m(x)$ is periodic for some $m \geq 0$.

If x is periodic, the minimum positive integer n such that $\phi^n(x) = x$ is called the **exact period** of x.

Note: x is preperiodic if and only if its forward orbit is finite.

Today:

$$X = \mathbb{P}^1(\mathbb{C}) \qquad \text{and} \qquad \phi \in \mathbb{C}(z)$$

or

$$X = \mathbb{P}^1(\mathbb{C}_p) \qquad \text{and} \qquad \phi \in \mathbb{C}_p(z)$$

Dynamics on $\mathbb{P}^1(\mathbb{C}_v)$

Classifying Periodic Points:

If $x \in \mathbb{P}^1$ is periodic of exact period n, then $\lambda := (\phi^n)'(x)$ is the **multiplier** of x. We say x is

- **attracting** if $|\lambda|_v < 1$.
- **repelling** if $|\lambda|_v > 1$.
- **neutral** (or **indifferent**) if $|\lambda|_v = 1$.

Note:

(1) The multiplier is the the same for all points in the periodic cycle of x.
(2) The multiplier is coordinate-independent.

The Spherical Metric

There is a spherical metric on $\mathbb{P}^1(\mathbb{C}_p)$ analogous to that on $\mathbb{P}^1(\mathbb{C})$:

$$\Delta(x, y) = \frac{|x - y|_p}{\max(1, |x|_p)\max(1, |y|_p)}$$

Definition. Write $\phi(z) \in \mathbb{C}_p(z)$ as $f(z)/g(z)$, with $f, g \in \mathcal{O}_p[z]$ relatively prime such that at least one coefficient of f or g has absolute value 1.

Let $d = \deg \phi = \max\{\deg f, \deg g\}$.

We say ϕ has **good reduction** (at p) if the reductions $\overline{f}, \overline{g} \in \overline{\mathbb{F}}_p[z]$ are still relatively prime and we still have $d = \max\{\deg \overline{f}, \deg \overline{g}\}$.

Otherwise, we say ϕ has **bad reduction** (at p).

If there is an LFT $f \in \mathrm{PGL}(2, \mathbb{C}_p)$ such that $f \circ \phi \circ f^{-1}$ has good reduction, we say ϕ has **potential good reduction**.

Example. Any monic polynomial ϕ in $\mathcal{O}_p[z]$ has good reduction.

Example. $\phi(z) = pz^3 + z$ has bad reduction, but $p^{1/2}\phi(p^{-1/2}z) = z^3 + z$ has good reduction; so ϕ has potential good reduction.

Example. $\phi(z) = z^2 + 1/4 = (4z^2 + 1)/4$ has bad reduction at $p = 2$. But $\phi(z + 1/2) - 1/2 = z^2 + z$ has good reduction; so ϕ has potential good reduction.

Example. $\phi(z) = \dfrac{3z^2 + 5z}{2z^2 - 7}$ has bad reduction at $p = 7$ and $p = 13$:

At $p = 7$, $\overline{\phi}(z) = \dfrac{3z(z+4)}{2z^2} = \dfrac{3(z+4)}{2z}$

At $p = 13$, $\overline{\phi}(z) = \dfrac{3z(z+6)}{2(z-6)(z+6)} = \dfrac{3z}{2(z-6)}$

The resultant $\mathrm{Res}(3z^2 + 5z, 2z^2 - 7)$ is $91 = 7 \cdot 13$, and so ϕ has good reduction at every other prime p.

Fatou and Julia Sets

We partition $\mathbb{P}^1$ into two subsets,
the **Fatou set** $\mathcal{F} = \mathcal{F}_\phi$ and the **Julia set** $\mathcal{J} = \mathcal{J}_\phi$.

Definitions

$$\begin{aligned}\mathcal{F} &= \{x \in \mathbb{P}^1 : \{\phi^n\}_{n\ge 0} \text{ equicontinuous on a neighborhood of } x\} \\ &= \{x \in \mathbb{P}^1 : \text{for } y \text{ near } x,\ \phi^n(y) \text{ is near } \phi^n(x)\} \\ &= \{\text{areas where small errors stay small}\}\end{aligned}$$

$$\begin{aligned}\mathcal{J} &= \mathbb{P}^1 \setminus \mathcal{F} \\ &= \{\text{areas where small errors may become large}\}\end{aligned}$$

For both $\mathbb{C}$ and $\mathbb{C}_p$:

(1) "near" with respect to the spherical metric.
(2) by definition, $\mathcal{F}$ is open, and $\mathcal{J}$ is closed.
(3) $\mathcal{F}_{\phi^n} = \mathcal{F}_\phi$, and $\mathcal{J}_{\phi^n} = \mathcal{J}_\phi$.
(4) $\phi(\mathcal{F}) = \mathcal{F}$, and $\phi(\mathcal{J}) = \mathcal{J}$.
(5) All attracting periodic points are Fatou.
(6) All repelling periodic points are Julia.

Definition.
If $\phi(z) \in \mathbb{C}_p[z]$ or $\mathbb{C}[z]$ is a polynomial of degree $d \geq 2$, the **filled Julia set** of ϕ is

$$\mathcal{K} = \mathcal{K}_\phi = \{x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n \geq 1} \text{ is bounded}\}.$$

Fact: $\partial\mathcal{K} = \mathcal{J}$.

Example. $\phi(z) = z^2 + c \in \mathbb{C}_p[z]$

- If $|4c|_p \leq 1$, then $\mathcal{K} = \overline{D}(a, 1)$ and $\mathcal{J} = \varnothing$
- If $|4c|_p > 1$, then $\mathcal{K} = \mathcal{J}$ is a Cantor set, contained in $\mathbb{Q}_p(\sqrt{1-4c})$

  (I.e., if $p \geq 3$, contained in $\mathbb{Q}_p(\sqrt{c})$.)

Example. (Smart and Woodcock). $\phi(z) = \dfrac{z^p - z}{p}$

Then $\mathcal{K} = \mathcal{J} = \mathbb{Z}_p$.

Example. (Morton, Silverman). Any ϕ of good reduction (or potentially good reduction) has empty Julia set.

Example. $p = 2$, $\phi(z) = z^8 + \frac{1}{2}z^4$.
Then ϕ has bad reduction, even after a change of coordinate.

But $\mathcal{J} = \varnothing$.

Example. $p \geq 3$, $\phi(z) = z^2(z - a)^p$.

- If $|a|_p \leq 1$, then ϕ has good reduction.
- If $1 < |a|_p \leq |p|_p^{-p/(2p+2)}$, then ϕ has bad reduction in all coordinates, but $\mathcal{J} = \varnothing$.
- For any $N \geq 1$, it is possible to choose $|a|_p > |p|_p^{-p/(2p+2)}$ so that:

(1) ϕ has no repelling periodic points of period $\leq N$, but

(2) ϕ has a repelling periodic point of period $N + 1$.

Contrasts

$\mathbb{C}$	$\mathbb{C}_p$
Some indifferent points of ϕ are Fatou, and some are Julia.	**All** indifferent points of ϕ are Fatou.
If D is a disk and $\bigcup_n \phi^n(D)$ omits at least 3 points of $\mathbb{P}^1(\mathbb{C})$, then $D \subseteq \mathcal{F}$. (Montel)	If D is a disk and $\bigcup_n \phi^n(D)$ omits at least 2 points of $\mathbb{P}^1(\mathbb{C}_p)$, then $D \subseteq \mathcal{F}$. (Hsia)
$\mathcal{J}$ is compact.	$\mathcal{J}$ may not be compact.
$\mathcal{J}$ is nonempty if $\deg(\phi) \geq 2$.	$\mathcal{J}$ may be empty for arbitrary degree.
$\mathcal{F}$ may be empty.	$\mathcal{F}$ is nonempty.
$\mathcal{J}$ is the closure of the set of repelling periodic points.	??? (various partial results) (Hsia, Bézivin)

Components of $\mathcal{F}$ in $\mathbb{P}^1(\mathbb{C})$

Let $\mathcal{C} = \{\text{connected components of } \mathcal{F}\}$.
Then ϕ induces

$$\Phi : \mathcal{C} \to \mathcal{C}$$

by $\Phi(U) = \phi(U)$.

Theorem. (Fatou, 1910s; Shishikura, 1987)
If $\phi \in \mathbb{C}(z)$ has degree $d \geq 2$, then there are at most $2d - 2$ periodic cycles of components of $\mathcal{F}_\phi$.

Fatou used classical analysis to get a bound of $6d - 6$. Shishikura used quasiconformal surgery to sharpen the bound.

Bonus. Fatou also classified the possible dynamics on a periodic component up to conjugacy.

Theorem. (Sullivan, 1985)
If $\phi \in \mathbb{C}(z)$, then $\mathcal{F}_\phi$ has no wandering domains. That is, all Fatou components are preperiodic.

The proof uses quasiconformal maps and moduli spaces of rational functions.

Components of $\mathcal{F}$ in $\mathbb{P}^1(\mathbb{C}_p)$

How can we define a "connected component" of the Fatou set?

($\mathbb{C}_p$ is totally disconnected.)

Recall.
A **disk** in $\mathbb{P}^1(\mathbb{C}_p)$ is either a disk in $\mathbb{C}_p$ or the complement in $\mathbb{P}^1(\mathbb{C}_p)$ of a disk.

A **connected affinoid** is a nonempty intersection of finitely many $\mathbb{P}^1(\mathbb{C}_p)$-disks— i.e., a disk with finitely many subdisks removed.

Idea:
Disks and connected affinoids act like connected sets.

If U is a disk (or connected affinoid), then so is $\phi(U)$. [OR else $\phi(U) = \mathbb{P}^1(\mathbb{C}_p)$.]

Definition. $U \subseteq \mathbb{P}^1(\mathbb{C}_p)$ is **dynamically stable** if $\bigcup_n \phi^n(U)$ omits at least three points of $\mathbb{P}^1(\mathbb{C}_p)$.

Definition.
Let $\mathcal{B}$ be the set of all dynamically stable:
(1) open or arbitrary
(2) disks or connected affinoids.

Note: For $U, V \in \mathcal{B}$,

- If $\phi(U) \neq \mathbb{P}^1(\mathbb{C}_p)$, then $\phi(U) \in \mathcal{B}$.
- If $U \cap V \neq \varnothing$ and $U \cup V \neq \mathbb{P}^1(\mathbb{C}_p)$, then $U \cup V \in \mathcal{B}$ and $U \cap V \in \mathcal{B}$.

Definition.
Let $x \in \mathcal{F}$. The **$\mathcal{B}$-component** of $\mathcal{F}$ containing x is the union of all $U \in \mathcal{B}$ with $x \in U \subseteq \mathcal{F}$.

Note: The definition is equivalent to:

$x, y \in \mathcal{F}$ are in the same $\mathcal{B}$-component of $\mathcal{F}$ if there are $U_0, \ldots, U_n \in \mathcal{B}$ with
(1) $U_i \subseteq \mathcal{F}$ for all $i = 0, \ldots, n$,
(2) $U_i \cap U_{i+1} \neq \varnothing$ for all $i = 1, \ldots, n$,
(3) $x \in U_0$ and $y \in U_n$.

Note:
If $\mathcal{B}$ is one of the two classes of disks, then all $\mathcal{B}$-components are either disks or all of $\mathbb{P}^1(\mathbb{C}_p)$.

But if $\mathcal{B}$ is one of the two classes of affinoids, then $\mathcal{B}$-components can be more complicated.
(They can still be just disks or affinoids, though.)

Note: $\mathcal{B} = \{$open dynamically stable affinoids$\}$ gives components related to Berkovich components.

Side note:

- For disks, we can weaken "dynamically stable" to omitting only two points, instead of three.
- If $\mathcal{J} \neq \varnothing$, we can remove the "dynamically stable" condition.

In several of my papers, I defined:

D-components:
$\mathcal{B} = \{$arbitrary connected disks$\}$.

Analytic components:
$\mathcal{B} = \{$arbitrary connected affinoids$\}$.

[without the dynamically stable condition].

Let $\mathcal{C} = \{\mathcal{B}\text{-components of } \mathcal{F}_\phi\}$.

Define

$$\Phi : \mathcal{C} \to \mathcal{C}$$

by

$$\Phi(U) = \text{the component containing } \phi(U)$$

If $\mathcal{B}$ is one of the two classes of affinoids, then $\Phi(U) = \phi(U)$.

If $\mathcal{B}$ is one of the two classes of disks, then it is possible that $\Phi(U) \supsetneq \phi(U)$.

However, even in that case, there are at most $d-1$ components V such that $\phi(U) \subsetneq V$ for some component U.

Example. If ϕ has good reduction, then the components are the residue classes of $\mathbb{P}^1(\mathbb{C}_p)$.

[True for all four component types.]

Example. (Hsia)

$p > 3$, $\phi(z) = pz^3 + z^2 + 1 \in \mathbb{C}_p[z]$.

- For any $\varepsilon > 0$, there is a repelling periodic point x with $1 < |x|_p < 1 + \varepsilon$.
- $\mathcal{J}$ is noncompact.
- Each open disk $D(a, 1)$ for $|a|_p < 1$ is a preperiodic component for both open component types.

  Infinitely many such disks are periodic and contain infinitely many periodic points.
- The closed disk $\overline{D}(0, 1)$ is a fixed component for both of the other component types.
- All Fatou components either eventually land in $\overline{D}(0, 1)$ or else are attracted to ∞.

Polynomials

For $\phi \in \mathbb{C}_p[z]$ with $\deg \phi \geq 2$, let

- $\mathcal{K}$ be the filled Julia set
- $U_0 \subseteq \mathbb{C}_p$ be the smallest disk containing $\mathcal{K}$
  [**Fact:** U_0 is a closed disk]
- $V = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathcal{K}$
- $W = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus U_0$.

Recall $\partial\mathcal{K} = \mathcal{J}$.

The component at ∞:

(1) For both affinoid component types, V is a fixed component of the Fatou set.

(2) For both disk component types, W is a fixed component of the Fatou set.

In that case, if $\mathcal{K} \subsetneq U_0$, then there are infinitely many components W' such that $\phi(W') \subsetneq W$.

Finite components:

(1) For both open component types, the finite Fatou components are all open disks.

(2) For both of the other component types, the finite Fatou components are all closed disks.

It is possible to have infinitely many **periodic** components.

Example. Let $p \geq 3$ be any odd prime, and let

$$\phi(z) = z^2 + \frac{pz^2}{z-1} \in \mathbb{C}_p(z).$$

Then $\mathcal{F}_\phi$ has infinitely many periodic components, including open disks of the form $D(\alpha, 1)$ for various $|\alpha|_p = 1$.

(They come from the infinitely many periodic points of the squaring map over $\overline{\mathbb{F}}_p$.)

(In Berkovich language, ϕ fixes the Gauss point of Berkovich space with degree 2.)

Example. Let $p = 2$, and let

$$\phi(z) = \frac{z^3 + 2z}{z+4} \in \mathbb{C}_2(z).$$

Then $\mathcal{F}_\phi$ has infinitely many periodic components, each containing infinitely many attracting periodic points.

Furthermore, there is a sequence of such components accumulating at the repelling fixed point $0 \in \mathcal{J}_\phi$.

Questions about components.

(1) Is there any control over the number of periodic cycles of components?

(2) Are there No Wandering Domains? (No p-adic theory of quasi-conformal maps.)

(3) Can dynamics on a fixed (or periodic) component be classified?

Question (3) is mostly answered.

Questions (1) and (2) have partial answers:

Theorem. (RB, 1998) Let $K/\mathbb{Q}_p$ be a finite extension and let $\phi(z) \in K(z)$. (I.e., $\phi(z) \in \overline{\mathbb{Q}}_p(z)$.)

Suppose $\mathcal{J}_\phi$ contains **no wild recurrent critical points**. (NWRJCP hypothesis.)

Then $\mathcal{F}_\phi$ has no wandering components. In addition, only finitely many periodic components of $\mathcal{F}_\phi$ contain K-points.

[True for any of the four component types.]

Definition. A point $x \in \mathbb{P}^1(\mathbb{C}_p)$ is *recurrent* if the forward iterates $\{\phi^n(x)\}_{n\geq 1}$ accumulate at x.

Definition. A critical point $x \in \mathbb{P}^1(\mathbb{C}_p)$ of ϕ is *wild* if p divides the order of ramification of ϕ at x.

(So 0 is a wild critical point of $\phi(z) = z^{np}$.)

The proof uses a bounded distortion argument.

Related Results:

(RB, 2002):
There is $a \in \mathbb{C}_p$ such that

$$\phi_a(z) = (1-a)z^{p+1} + az^p \in \mathbb{C}_p[z]$$

has no critical points in the Julia set but **does** have wandering domains.

(RB, 2003):
The set of parameters for which ϕ_a has a wandering domain is dense in $\{|a| > 1\}$.

(Rivera-Letelier, 2004):
There are polynomials

$$\psi_b(z) = -p^{-p}z(z-1)^p + b \in \mathbb{Q}_p[z]$$

that fail the NWRJCP hypothesis.

Dynamics on a fixed component.

For open affinoid components, Rivera-Letelier (2000) has classified dynamics on a fixed component U into two broad types:

Attracting:

U may have a complicated shape.

There is a single attracting fixed point, and all points are U are attracted to it.

Quasiperiodic:

U is a rational open connected affinoid

(i.e., rational open disk minus finite number of rational closed disks).

Any (finite) number of holes is possible.

There are **infinitely many** indifferent periodic points.

Even for a given number of holes, there appear to be **many** non-conjugate dynamical types.