(Non)-Uniform Bounds for Rational Preperiodic Points in Arithmetic Dynamics

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Dynamics over Number Fields

K = global field, $N \geq 1$.

$\phi : \mathbb{P}^N(K) \to \mathbb{P}^N(K)$ morphism over K, degree $d \geq 2$.
($N = 1$: $\phi \in K(z)$ is a rational function.)

$\mathrm{Preper}(\phi, K) := \{\text{preperiodic points of } \phi \text{ in } \mathbb{P}^1(K)\}$.

Example. $\phi(z) = z^2 - \dfrac{133}{144}$, on $\mathbb{P}^1(\mathbb{Q})$.

$$0 \mapsto -\frac{133}{144} \mapsto -\frac{1463}{20736} \mapsto -\frac{394995503}{429981696} \mapsto \cdots$$

$$0 \mapsto -\frac{(*)}{2^4 \cdot 3^2} \mapsto \frac{(*)}{2^8 \cdot 3^4} \mapsto \frac{(*)}{2^{16} \cdot 3^8} \mapsto \cdots$$

$$\frac{17}{12} \mapsto \frac{13}{12} \mapsto \frac{1}{4} \mapsto -\frac{31}{36} \mapsto -\frac{59}{324} \mapsto \cdots$$

$$\frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3} \mapsto \frac{(*)}{2^2 \cdot 3^0} \mapsto \frac{(*)}{2^2 \cdot 3^2} \mapsto \frac{(*)}{2^2 \cdot 3^4} \mapsto \cdots$$

$$\frac{43}{12} \mapsto \frac{143}{12} \mapsto \frac{1693}{12} \mapsto \frac{238843}{12} \mapsto \frac{4753831543}{12} \mapsto \cdots$$

$$\phi(z) = z^2 - \frac{133}{144}.$$

$$\frac{1}{12} \longmapsto -\frac{11}{12} \leftrightarrows -\frac{1}{12} \longleftarrow \frac{11}{12}$$

$$\frac{7}{12} \longmapsto -\frac{7}{12} \longmapsto -\frac{7}{12}$$

$$-\frac{19}{12} \longmapsto \frac{19}{12} \longmapsto \frac{19}{12}$$

$$\infty \mapsto \infty$$

$$\phi(z) = z^2 - \frac{29}{16}.$$

$$\begin{array}{ccccccc}
 & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & &
\end{array}$$

$$\infty \mapsto \infty$$

Theorem (Northcott, 1950): Let K be a global field. Let $\phi : \mathbb{P}^N(K) \to \mathbb{P}^N(K)$ be a morphism, defined over K, of degree $d \geq 2$. Then

$$\#\mathrm{Preper}(\phi, K) < \infty.$$

Dynamical Uniform Boundedness Conjecture (Morton & Silverman, 1994):
For any integers $d \geq 2$, $D \geq 1$, and $N \geq 1$, there is a constant $C = C(d, D, N)$ such that

- for any number field K with $[K : \mathbb{Q}] = D$, and
- for any morphism $\phi : \mathbb{P}^N(K) \to \mathbb{P}^N(K)$ defined over K and of degree d,

$$\#\mathrm{Preper}(\phi, K) \leq C(d, D, N).$$

Conjecture (DUBC Lite):
There is a constant $C > 0$ so that for any **quadratic polynomial** $\phi \in \mathbb{Q}[z]$,

$$\#\mathrm{Preper}(\phi, \mathbb{Q}) \leq C.$$

Refined DUBC Lite Conjecture (Poonen, 1998):
The DUBC Lite Constant is 9.

Definition.
Let K be a global field, $v \in M_K$ non-archimedean, and $\phi \in K(z)$ a rational function.

We say ϕ has **good reduction** at v if ϕ may be written in homogeneous coordinates as $[f, g]$, i.e.,

$$\phi\left(\frac{x}{y}\right) = \frac{f(x,y)}{g(x,y)}$$

for some $f, g \in \mathcal{O}_v[x, y]$ homogeneous of the same degree such that:

> The reductions $\overline{f}$ and $\overline{g}$ modulo v
> have no common zeros besides $(0, 0)$.

In that case, ϕ maps residue classes into residue classes.

Idea: ϕ still "makes sense" everywhere modulo v.

Example. Any polynomial with v-adic integer coefficients and with $|a_d|_v = 1$ has good reduction:

$$\begin{aligned}\phi(z) &= a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0 \\ &= \frac{a_d x^d + a_{d-1} x^{d-1} y + \cdots + a_0 y^d}{y^d}.\end{aligned}$$

Example. $\phi(z) = z^2 - \dfrac{133}{144} \in \mathbb{Q}[z]$
has bad reduction at $v = 2, 3, \infty$.

Modulo 2 or 3:

$$\phi(z) = \frac{144x^2 - 133y^2}{144y^2} \equiv \frac{-y^2}{0}.$$

Example. $\psi(z) = \dfrac{3z^2 - 8z}{4z + 11} \in \mathbb{Q}(z)$
has bad reduction at $v = 3, 5, 11, 13, \infty$.
[Resultant of $3z^2 - 8z$ and $4z + 11$ is $3 \cdot 5 \cdot 11 \cdot 13$.]

E.g. modulo 5:

$$\psi(z) = \frac{3x^2 - 8xy}{4xy + 11y^2} \equiv \frac{3x^2 - 3xy}{-xy + y^2} = \frac{3x(x - y)}{-y(x - y)}.$$

Theorem. (Pezda, Morton & Silverman, Zieve, 1990s). Let $\phi \in K(z)$ such that no iterate of ϕ is the identity, and with good reduction at v.
Let $x \in \mathbb{P}^1(K)$ be a K-rational **periodic** point of ϕ.

Then the period n of x is at most $O(N\mathfrak{p}_v^2)$.

[$N\mathfrak{p}_v$ is the norm of the prime ideal associated to v.]

Idea of Proof: Write $q = \#k_v = p^f$. So $N\mathfrak{p}_v = qp^e$.
WLOG $x = 0$.

$D(0,1)$ is periodic of period $m \leq \#\mathbb{P}^1(k_v) = q + 1$.

Let $\lambda = (\phi^m)'(0)$.
Good reduction implies $|\lambda|_v \leq 1$. If $<$, then $n = m$.

Otherwise, let r be the order of $\overline{\lambda} \in k_v^\times$.
Note $r \leq \#k_v^\times = q - 1$.

Then $n = mrp^E$ for some integer $E \geq 0$.
Pezda/Zieve: $E \leq e$. [In fact, **much** less.]

Note:

- There is no reference to the degree $d = \deg \phi$!!!
- The proof works entirely in the local field K_v.
- More work gives $\#\text{Preper}(\phi, \mathbb{Q}) < [\approx d^{N\mathfrak{p}^2}]$.

Definition. Let $v \in M_K$, and let $\phi \in K[z]$ be a polynomial of degree $d \geq 2$. Let $\mathbb{C}_v$ be the completion of an algebraic closure of K_v.

The *filled Julia set* of ϕ at v is

$$\mathcal{K}_{\phi,v} := \{x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n\geq 0} \text{ is bounded}\}$$

Note:

(1) All preperiodic points (besides ∞) lie in $\mathcal{K}_{\phi,v}$.

(2) If ϕ is good at v, then $\mathcal{K}_{\phi,v} = \overline{D}(0,1)$.

Theorem. (Call & Goldstine, 1997.) Let $c \in \mathbb{Q}$ and let $\phi(z) = z^2 + c$. Let s be the number of bad primes (i.e., one plus the number of **distinct** primes dividing the denominator of c).
Then

$$\#\mathrm{Preper}(\phi, \mathbb{Q}) \leq 1 + 2^{s+2} = O(2^s)$$

except for $c = -2$, with $\#\mathrm{Preper}(\phi, \mathbb{Q}) = 6$.

Idea of Proof:
1. (p-adic dynamics step):

(a.) For good primes p, we know $\mathcal{K}_{\phi,p} = \overline{D}(0,1)$ is disk of radius 1.

(b.) For bad primes p, prove that $\mathcal{K}_{\phi,p}$ sits inside a union of two disks of radius 1.
(Slightly different for $p = 2, \infty$.)

2. (global step):

In each choice of one unit disk at each prime (or interval length 1 at $v = \infty$), there is only one rational number.

Theorem. (RB, 2004.) Let K be a global field, and let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Let s be the number of bad primes (i.e, **not potentially good**) of ϕ. Then

$$\#\mathrm{Preper}(\phi, K) \leq O\left(\frac{d^2}{\log d} \cdot s \log s\right).$$

More properties of $\mathcal{K}_{\phi,v}$:

(1) If ϕ is monic, then the smallest disk $U_0 := \overline{D}(a, r)$ containing $\mathcal{K}_{\phi,v}$ has radius $r \geq 1$.

(2) If v is non-archimedean, $r = 1 \iff$ potentially good.

(3) Define $U_n := \phi^{-n}(U_0)$. Then

$$U_0 \supset U_1 \supset U_2 \supset \cdots$$

and

$$\mathcal{K}_{\phi,v} = \bigcap_{n \geq 0} U_n.$$

(4) If v is non-archimedean, then U_n is a disjoint union of closed disks.

Lemma 1. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$. Assume ϕ is monic, and let $r_{\phi,v}$ be the radius of the smallest disk $U_0 \subset \mathbb{C}_v$ containing $\mathcal{K}_{\phi,v}$.
Given $N \geq 2$, let $x_1, \ldots, x_N \in \mathcal{K}_{\phi,v}$. Then

$$\prod_{i \neq j} |x_i - x_j|_v \leq B_v(N) \cdot r_{\phi,v}^{(d-1)N \log_d N},$$

where

$$B_v(N) = \begin{cases} N^N & \text{if } v \text{ is archimedean,} \\ 1 & \text{if } v \text{ is non-archimedean.} \end{cases}$$

Note:

(1) If ϕ not monic, you also get a correction factor of $|a_d|^{-N(N-1)/(d-1)}$ on the right.

(2) The N^N can probably be substantially reduced (but not eliminated).

(3) Otherwise, these bounds are sharp: $\phi(z) = z^d + c$.

Proof.

Let $U_0 = \overline{D}(a, r_{\phi,v})$ be the smallest disk containing $\mathcal{K}_{\phi,v}$.

For any integer $j \geq 0$, write

$$j = c_0 + c_1 d + c_2 d^2 + \cdots + c_M d^M$$

in base d. $(0 \leq c_i \leq d - 1.)$ Let

$$f_j(z) = \prod_{i=0}^{M} [\phi^i(z) - a]^{c_i},$$

so that f_j is a monic polynomial of degree j with

$$|f_j(x)|_v \leq r_{\phi,v}^{c_0+c_1+c_2+\cdots+c_M}$$

for $x \in \mathcal{K}_{\phi,v}$.

Meanwhile, $\prod_{i \neq j} (x_i - x_j) = \pm(\det V)^2$, where

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \ldots & x_N^{N-1} \end{bmatrix}.$$

Column operations (f_j monic) gives

$$\det V = \det A,$$

where

$$A = \begin{bmatrix} 1 & f_1(x_1) & f_2(x_1) & \dots & f_{N-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \dots & f_{N-1}(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(x_N) & f_2(x_N) & \dots & f_{N-1}(x_N) \end{bmatrix}.$$

Hadamard: $|\det A|_v \leq \prod_{j=0}^{N-1} \|(j^{\text{th}} \text{ column})\|$

(v arch: L^2-norm; v non-arch: L^∞-norm.)

But by properties of f_j:

$$\|(j^{\text{th}} \text{ column})\| \leq b_v(N) r_{\phi,v}^{c_0+\dots+c_M}$$

where $b_v(N) = \begin{cases} 1 & v \text{ non-archimedean,} \\ \sqrt{N} & v \text{ archimedean.} \end{cases}$

Hence

$$\prod_{i \neq j} |x_i - x_j|_v = |\det A|_v^2 \leq B_v(N) \prod_{j=0}^{N-1} r_{\phi,v}^{2(c_0+\dots+c_M)}.$$

That is,

$$\prod_{i \neq j} |x_i - x_j|_v \leq B_v(N) \cdot r_{\phi,v}^{E(N,d)},$$

where

$$\begin{aligned} E(N,d) &= 2 \sum_{j=0}^{N-1} [c_0(j) + c_1(j) + \cdots + c_M(j)] \\ &= \text{twice the sum of all base-}d\text{ coefficients} \\ &\quad \text{of all integers from } 0 \text{ to } N-1. \end{aligned}$$

Finally, it is elementary to show that

$$E(N,d) \leq (d-1) N \log_d N.$$

Lemma 2. Let K, v, ϕ, d, and $r_{\phi,v}$ be as in Lemma 1. Assume that

$$r_{\phi,v} > \begin{cases} 4d & \text{if } v \text{ is archimedean,} \\ 1 & \text{if } v \text{ is non-archimedean.} \end{cases}$$

Then there are disjoint sets $V_1, V_2 \subseteq \mathbb{C}_v$ and an integer $1 \le m \le d-1$ such that

- $\mathcal{K}_{\phi,v} = V_1 \cup V_2$,
- $\phi : V_1 \twoheadrightarrow \mathcal{K}_{\phi,v}$ is m-to-1,
- $\phi : V_2 \twoheadrightarrow \mathcal{K}_{\phi,v}$ is $(d-m)$-to-1, and
- For any $y_1 \in V_1$ and $y_2 \in V_2$,

$$|y_1 - y_2|_v \ge r_{\phi,v} \cdot \begin{cases} 2/d & \text{if } v \text{ is archimedean,} \\ 1 & \text{if } v \text{ is non-archimedean.} \end{cases}$$

Idea: If the diameter of $\mathcal{K}_{\phi,v}$ is big, then it splits into (at least) two pieces which are very far apart.

Lemma 3. Under the hypotheses of Lemma 2, for any $x_1, \ldots, x_N \in V_1$,

$$\prod_{i \neq j} |x_i - x_j|_v \leq B'_v(N) r_{\phi,v}^{(d-1)N[\log_d N - P_m(N)]},$$

where

$$P_m(N) = \frac{d-m}{m(d-1)} N - (1 - \log_d m)$$

and

$$B'_v(N) = \begin{cases} N^N (d-1)^{P_m(N)} & \text{if } v \text{ is archimedean,} \\ 1 & \text{if } v \text{ is non-archimedean.} \end{cases}$$

- Similar bound for V_2, with $m \leftrightarrow d - m$.
- Throw in the same correction factor $|a_d|_v^{-N(N-1)/(d-1)}$ if ϕ is not monic.

Summary of the three Lemmas:

Lemma 1. For $x_1, \ldots, x_N \in \mathcal{K}_{\phi,v}$,

$$\prod_{i \neq j} |x_i - x_j|_v \leq r_v^{N \log N},$$

essentially.

That is:
On average, $|x_i - x_j|_v$ is only a little bigger than 1.

Lemma 2. If r_v is big enough, then $\mathcal{K}_{\phi,v}$ splits into two small pieces, V_1 and V_2, that are far apart.

Lemma 3. For $x_1, \ldots, x_N \in V_1$ (or in V_2),

$$\prod_{i \neq j} |x_i - x_j|_v \leq r_v^{-a_m N^2},$$

essentially.

That is:
On average, $|x_i - x_j|_v \approx r_v^{-a_m}$, which is small.

Theorem: Sketch of Proof.

[For simplicity, assume ϕ is monic.]

For each $v \in M_K$, let $R_v = r_{\phi,v}^{n_v}$.

[Actually, adjust R_v slightly at archimedean v.]

Let $w \in M_K$ be the place for which R_w is largest.
Split $\mathcal{K}_{\phi,w}$ into V_1 and V_2.

If V_1 contains N distinct rational preperiodic points

$$x_1, \dots, x_N \in V_1 \subseteq \mathbb{C}_w,$$

then by the product formula,

$$1 = \prod_{i \neq j} \prod_{v \in M_K} |x_i - x_j|_v^{n_v} \leq \prod_{v \text{ bad}} \prod_{i \neq j} |x_i - x_j|_v^{n_v}$$

$$\leq \left[R_w^{-P_m(N)} \prod_{v \text{ bad}} R_v^{\log_d N} \right]^{(d-1)N} \cdot \prod_{v \text{ arch}} (B_v \text{ or } B'_v)$$

$$\leq \left[R_w^{s \log_d N - P_m(N)} \right]^{(d-1)N} \cdot \prod_{v \text{ arch}} (B_v \text{ or } B'_v)$$

Since $R_w > 1$, we only need to choose N large enough so that

$$s \log_d N - \frac{d-m}{m(d-1)} N + (1 - \log_d m) < 0$$

to get a contradiction.

Letting N be slightly bigger than

$$N_m = \frac{m(d-1)}{d-m} s \log_d s$$

does the trick.

Do the same for V_2. So if the **total** number of rational preperiodic points is at least $N_m + N_{d-m}$, we get a contradiction.

The worst case is $m = 1$, which gives a total number of points on the order of at most

$$[1 + (d-1)^2] s \log_d s = (d^2 - 2d + 2) s \log_d s.$$

Heights

Let K be a number field.

The **standard height** on $\mathbb{P}^N(K)$ is

$$h([x_0,\dots,x_N]) := \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K}[K_v:\mathbb{Q}_v]\log\max\{|x_0|_v,\dots,|x_N|_v\}.$$

(Analogous definition for function fields.)

For $K=\mathbb{Q}$ and for $x_i\in\mathbb{Z}$ with $\gcd(x_0,\dots,x_N)=1$, we can write

$$h([x_0,\dots,x_N]) = \log\max\{|x_0|_\infty,\dots,|x_N|_\infty\}.$$

Key properties:

- If $\phi:\mathbb{P}^N\to\mathbb{P}^N$ is any morphism of degree d, then $h(\phi(x))-d\cdot h(x)$ is a bounded function of x.
- (**Non-degeneracy**) For global fields K and any real number B, the set of K-points of height at most B is finite.

Canonical Heights

Given a morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ defined over K of degree $d \geq 2$, the **canonical height** for ϕ on $\mathbb{P}^N(K)$ is

$$\hat{h}_\phi(x) = \lim_{n\to\infty} \frac{1}{d^n} h\left(\phi^n(x)\right).$$

We have:

- The limit converges.
- $\hat{h}_\phi - h$ is bounded.
- $\hat{h}_\phi(\phi(x)) = d \cdot \hat{h}_\phi(x)$.
- $\hat{h}_\phi(x) \geq 0$.
- For $N = 1$, ϕ a polynomial, and $x \neq \infty$:

  $$\hat{h}_\phi(x) = 0 \iff x \in \mathcal{K}_{\phi,v} \text{ for all } v \in M_K.$$
- If x is preperiodic, then $\hat{h}(x) = 0$.
- For global fields, if $\hat{h}(x) = 0$, then x is preperiodic.

Points of Small Canonical Height

a.k.a. "Almost" Preperiodic Points

Example. $\phi(z) = z^2 - \dfrac{181}{144}$

$$\frac{7}{12} \mapsto -\frac{11}{12} \mapsto -\frac{5}{12} \mapsto -\frac{13}{12} \mapsto -\frac{1}{12}$$

$$\mapsto -\frac{5}{4} \mapsto \frac{11}{36} \mapsto -\frac{377}{324} \mapsto \cdots$$

$\hat{h}_\phi(7/12) = 2^{-5}\log 3 = 0.03433\ldots$, vs.
$h(\phi) = h(181/144) = \log 181 = 5.198\ldots$
Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.00660\ldots$

Example. $\phi(z) = z^2 - \dfrac{36989}{19600}$

$$\frac{153}{140} \mapsto -\frac{97}{140} \mapsto -\frac{197}{140} \mapsto \frac{13}{140} \mapsto -\frac{263}{140}$$

$$\mapsto \frac{1609}{980} \mapsto \frac{38821}{48020} \mapsto \cdots$$

$\hat{h}_\phi(153/140) = 2^{-10}\log 5 + 2^{-4}\log 7 = 0.12319\ldots$, vs.
$h(\phi) = h(36989/19600) = \log 36989 = 10.518\ldots$
Ratio is $\hat{h}_\phi(153/140)/h(\phi) = 0.0117\ldots$

Another Point of Small Canonical Height

Example. $\phi(z) = -\frac{1}{24}z^3 + \frac{97}{24}z + 5$

$$-7 \mapsto 19 \mapsto -1 \mapsto 1 \mapsto 9$$

$$\mapsto 11 \mapsto -6 \mapsto -\frac{41}{4} \mapsto \frac{4323}{512} \mapsto \cdots$$

$\hat{h}_\phi(-7) = 0.0011\ldots$, vs.

$h(\phi) = \log(97) = 4.57\ldots$

Ratio is $\hat{h}_\phi(-7)/h(\phi) = 0.00025\ldots$

Conjecture. (Silverman)
Let K be a number field and $d \geq 2$.
There is a constant $C = C(K,d)$ such that if $\phi \in K(z)$ with $\deg\phi = d$, then for any non-preperiodic $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$