Rank Theorem and Trace Homework

Due Friday April 3

1. Prove the Rank Theorem, $\dim(C(\mathbf{A}))+\dim(N(\mathbf{A}))=n$ for any mxn matrix **A,** by using a basis argument like that used in lecture to prove dim(U+W)=dimU+dimW-dim(U∩W). Do not use a leading one/RREF argument. Hint: Take any basis $\{\mathbf{x}_1,\ldots,\mathbf{x}_p\}$ for $N(\mathbf{A})$, extend to a basis $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ for $\mathbf{C}^n$, and then think about the set $\{\mathbf{Ax}_{p+1},\ldots,\mathbf{Ax}_n\}$.

2. Let **A** be an mxn matrix. Prove that **A** and $\mathbf{A}^t\mathbf{A}$ have the same null space. Hint: use the facts that $\|\mathbf{Ax}\|^2 = (\mathbf{Ax})^t\mathbf{Ax}$ and $\|\mathbf{y}\| = 0$ if and only if $\mathbf{y} = \mathbf{0}$.

3. Let **A** be an mxn matrix. Prove that **A** and $\mathbf{A}^t\mathbf{A}$ have the same rank. Hint: use the Rank Theorem.

4. Let **A** be an mxn matrix with rank$\mathbf{A}=n$. Prove that $\mathbf{A}^t\mathbf{A}$ is invertible. Hint: use the result in exercise 3.

5. Prove that the trace of **A** equals the trace of $\mathbf{A}^T$, for any nxn matrix **A**.

6. Prove that the tr(**AB**) = tr(**BA**), where **A** and **B** are nxn matrices. Hint: use the definition of matrix multiplication $(\mathbf{AB})_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.